Space-Time Methods for the Wave Equation

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17 December 2018

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Wave Equation

Consider

$$\begin{aligned} \partial_{tt} u - \Delta_x u &= 0 \quad \text{in } (\mathbb{R}^d \setminus \Gamma) \times (0, \infty), \\ u(\,\cdot\,, 0) &= 0 \quad \text{in } \mathbb{R}^d \setminus \Gamma, \\ \partial_t u(\,\cdot\,, 0) &= 0 \quad \text{in } \mathbb{R}^d \setminus \Gamma \end{aligned}$$

and Kirchhoff's formula

$$u = \tilde{\mathcal{V}} * [\![\gamma_1 u]\!] - \mathcal{W} * [\![\gamma_0 u]\!]$$

with

$$\begin{aligned} [\![\gamma_0 u]\!] &:= \gamma_0^{\mathrm{int}} u - \gamma_0^{\mathrm{ext}} u, \\ [\![\gamma_1 u]\!] &:= \gamma_1^{\mathrm{int}} u - \gamma_1^{\mathrm{ext}} u. \end{aligned}$$

Laplace Transform Method [Bamberger, Ha Duong 1986]

$$-\Delta_x u + \partial_{tt} u = 0 \text{ in } \Omega \times \mathbb{R}_+, \quad u|_{\Gamma\times\mathbb{R}_+} = g \quad \text{on } \Gamma \times \mathbb{R}_+, \quad \text{i.c.}$$

1. Apply Laplace transform to the wave equation and end up with the Helmholtz equation in the frequency domain for $\omega \in \mathbb{C}_0$:

$$-\Delta_x U(\omega) - \omega^2 U(\omega) = 0 \text{ in } \Omega, \quad \gamma_0^{\mathrm{int}} U(\omega) = G(\omega) \text{ on } \Gamma$$

2. Reformulate the Helmholtz equation via potentials for $\omega \in \mathbb{C}_0$:

$$\tilde{V}(\omega)\colon\ H^{-1/2}(\Gamma) \to H^1(\Omega), \quad U(\omega) = \tilde{V}(\omega)\Lambda(\omega)$$

3. Analyse the bie explicitly in terms of $\omega \in \mathbb{C}_{\sigma_0}$ for fixed $\sigma_0 > 0$:

$$\|V(\omega)\| \leq C(\sigma_0)\,|\omega|\,, \quad \left\|V(\omega)^{-1}\right\| \leq C(\sigma_0)\,|\omega|^2$$

4. Apply Paley-Wiener theorem and convolution theorem for the inverse transform to the time domain:

$$u = \tilde{\mathcal{V}} * \lambda := \mathcal{L}^{-1}(\omega \mapsto \tilde{V}(\omega)\Lambda(\omega))$$

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Standard Space-Time Sobolev Spaces

$$\mathcal{H}^r_\sigma(\mathbb{R}, H^\nu(\Gamma)) := \left\{ u \in \mathrm{LT}(H^\nu(\Gamma)) \mid \quad \|u\|_{\sigma,r,H^\nu(\Gamma)} < \infty \right\}$$

with $\sigma > 0$, $r \in \mathbb{R}$, $\nu \in \mathbb{R}$ and the norm

$$\|u\|^2_{\sigma,r,H^\nu(\Gamma)} := \int_{-\infty+\imath\sigma}^{\infty+\imath\sigma} |\omega|^{2r} \, \|\mathcal{L}\{u\}(\omega)\|^2_{H^\nu(\Gamma)} \mathrm{d}\omega.$$

For $r = k \in \mathbb{N}$ it holds

$$\|u\|^2_{\sigma,k,H^\nu(\Gamma)} = 2\pi \int_{\mathbb{R}} \mathrm{e}^{-2\sigma t} \left\| \frac{\mathrm{d}^k u}{\mathrm{d}t^k}(t) \right\|^2_{H^\nu(\Gamma)} \mathrm{d}t = \left\| \frac{\mathrm{d}^k u}{\mathrm{d}t^k} \right\|^2_{\sigma,0,H^\nu(\Gamma)}.$$

Energy Space-Time Sobolev Spaces

$$H^{r;\nu}_{\sigma,\Gamma} := \left\{ u \in \mathrm{LT}(H^\nu(\Gamma)) \mid \quad \|u\|_{\sigma,\Gamma;r;\nu} < \infty \right\}$$

with $\sigma > 0$, $r \in \mathbb{R}$, $\nu \in \mathbb{R}$ and the norm

$$\|u\|^2_{\sigma,\Gamma;r;\nu} := \int_{-\infty+\imath\sigma}^{\infty+\imath\sigma} |\omega|^{2r} \, \|\mathcal{L}\{u\}(\omega)\|^2_{\nu,\omega,\Gamma} \mathrm{d}\omega.$$

The ω-dependent norm $\|\cdot\|_{\nu,\omega,\Gamma}$ is defined analogously by

$$\|v\|^2_{\nu,\omega,\mathbb{R}^d} := \int_{\mathbb{R}^d} (|\omega|^2 + |\xi|^2)^\nu \, |\mathcal{F}\{v\}(\xi)|^2 \, \mathrm{d}\xi \quad \text{for } v \in H^\nu(\mathbb{R}^d).$$

It holds

$$\begin{aligned} \mathcal{H}^{r+\nu}_\sigma(\mathbb{R}, H^\nu(\Gamma)) \quad \subset \quad H^{r;\nu}_{\sigma,\Gamma} \quad \subset \quad \mathcal{H}^r_\sigma(\mathbb{R}, H^\nu(\Gamma)), \\ H^{r;\nu}_{\sigma,\Gamma} \quad = \quad \mathcal{H}^r_\sigma(\mathbb{R}, H^\nu(\Gamma)) \cap \mathcal{H}^{r+\nu}_\sigma(\mathbb{R}, L^2(\Gamma)). \end{aligned}$$

Boundedness of the Single Layer Operator

It holds for $\sigma \geq \sigma_0 > 0$ and $C = C(\Gamma, \sigma_0)$

$$\begin{aligned}
\|\mathcal{V} * \lambda\|_{\sigma,0,H^{1/2}(\Gamma)} &\leq C\|\lambda\|_{\sigma,1,H^{-1/2}(\Gamma)}, \\
\|\mathcal{V} * \lambda\|_{\sigma,\Gamma;0;1/2} &\leq C\|\lambda\|_{\sigma,\Gamma;1;-1/2}, \\
\left\|\mathcal{V}^{-1} * g\right\|_{\sigma,0,H^{-1/2}(\Gamma)} &\leq C\|g\|_{\sigma,2,H^{1/2}(\Gamma)}, \\
\left\|\mathcal{V}^{-1} * g\right\|_{\sigma,\Gamma;0;-1/2} &\leq C\|g\|_{\sigma,\Gamma;1;1/2}.
\end{aligned}$$

Loss of regularity in time

$$\begin{aligned}
&\mathcal{V}^{-1} * \mathcal{V}* : \mathcal{H}^3_\sigma(\mathbb{R}, H^{-1/2}(\Gamma)) \to \mathcal{H}^0_\sigma(\mathbb{R}, H^{-1/2}(\Gamma)), \\
&\mathcal{V}^{-1} * \mathcal{V}* : H^{2;-1/2}_{\sigma,\Gamma} \to H^{0;-1/2}_{\sigma,\Gamma}.
\end{aligned}$$

Coercivity in the Time-Domain

Bilinear form in the frequency domain for $\omega \in \{\operatorname{Im}\omega \geq \sigma_0 > 0\}$:

$$\langle \psi, -\imath\omega V(\omega)\varphi \rangle \quad \forall \varphi, \psi \in H^{-1/2}(\Gamma).$$

Define a bilinear form for $\sigma \geq \sigma_0 > 0$

$$a_\sigma(\lambda, \tau) := \int_0^\infty \mathrm{e}^{-2\sigma t} \left\langle \tau(t), \left(\mathcal{V} * \frac{\mathrm{d}\lambda}{\mathrm{d}t} \right)(t) \right\rangle_\Gamma \mathrm{d}t.$$

For $C = C(\Gamma, \sigma_0)$ there are the coercivity estimates

$$a_\sigma(\lambda,\lambda) \geq C \|\lambda\|^2_{\sigma,-1/2,H^{-1/2}(\Gamma)}, \qquad \lambda \in \mathcal{H}_\sigma^{-1/2}(\mathbb{R}, H^{-1/2}(\Gamma)),$$

$$a_\sigma(\lambda,\lambda) \geq C \|\lambda\|^2_{\sigma,\Gamma;0;-1/2}, \qquad \lambda \in H^{0;-1/2}_{\sigma,\Gamma}$$

and the boundedness estimates

$$|a_\sigma(\lambda,\tau)| \leq C \|\lambda\|_{\sigma,1,H^{-1/2}(\Gamma)} \|\tau\|_{\sigma,1,H^{-1/2}(\Gamma)},$$

$$|a_\sigma(\lambda,\tau)| \leq C \|\lambda\|_{\sigma,\Gamma;1;-1/2} \|\tau\|_{\sigma,\Gamma;1;-1/2}.$$

Variational Formulation

Let $g \in \mathcal{H}_\sigma^{3/2}(\mathbb{R}, H^{1/2}(\Gamma))$ be the given Dirichlet data.

Find $\lambda \in \mathcal{H}_\sigma^{1}(\mathbb{R}, H^{-1/2}(\Gamma))$ with $\mathcal{V} * \lambda \in \mathcal{H}_\sigma^{1}(\mathbb{R}, H^{1/2}(\Gamma))$ such that

$$a_\sigma(\lambda, \tau) = \int_0^\infty \mathrm{e}^{-2\sigma t} \left\langle \frac{\mathrm{d}g}{\mathrm{d}t}(t), \tau(t) \right\rangle_\Gamma \mathrm{d}t$$

for all $\tau \in \mathcal{H}_\sigma^{1}(\mathbb{R}, H^{-1/2}(\Gamma))$.

Find $\lambda_h \in V_h^{m_1, m_2} := V_{h_x}^{m_1} \otimes V_{h_t}^{m_2}$ such that

$$a_\sigma(\lambda_h, \tau_h) = \int_0^\infty \mathrm{e}^{-2\sigma t} \left\langle \frac{\mathrm{d}g}{\mathrm{d}t}(t), \tau_h(t) \right\rangle_\Gamma \mathrm{d}t$$

for all $\tau_h \in V_h^{m_1, m_2}$.

Error Estimates

For sufficiently regular λ it holds for $m_1 = 0$, $m_2 = 1$

$$\begin{aligned} \|\lambda - \lambda_h\|_{\sigma,-1/2,H^{-1/2}(\Gamma)} &= \mathcal{O}(h_x^{1/2} + h_x^{-1/2} h_t^{1/2}), \\ \|\lambda - \lambda_h\|_{\sigma,0,H^{0}(\Gamma)} &= \mathcal{O}(h_x^{-1} + h_t^{-1/2}) \end{aligned}$$

and

$$\begin{aligned} \|\lambda - \lambda_h\|_{\sigma,\Gamma;0;-1/2} &= \mathcal{O}(\max\{h_x^{-1/2}, h_t^{-1/2}\}(h_x^{2/3} + h_t^2)), \\ \|\lambda - \lambda_h\|_{\sigma,\Gamma;0;0} &= \mathcal{O}(\max\{h_x^{-1}, h_t^{-1}\}(h_x^{2/3} + h_t^2)). \end{aligned}$$

For $h_x \sim h_t$ in $d = 2$ with $\sigma = 0$ numerical experiments [Gläfke] for $m_1 = 0$, $m_2 = 0$ show

$$\|\lambda - \lambda_h\|_{L^2(0,T;L^2(\Gamma))} = \mathcal{O}(h_x + h_t).$$

Two-Dimensional Wave Equation [Aimi et al. 2009]

Let $\Gamma = \{(x,0) \in \mathbb{R}^2 :\ x \in (0,L)\}$ be an open arc and consider

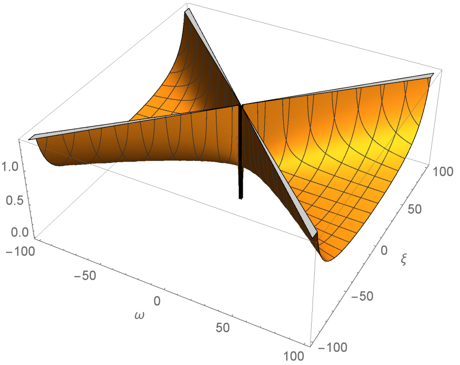
$$\begin{aligned} \partial_{tt} u - \Delta_x u &= 0 \quad \text{in } \mathbb{R}^2 \setminus \Gamma \times (0,T), \\ \partial_t u = u &= 0 \quad \text{in } \mathbb{R}^2 \setminus \Gamma \times \{0\}, \\ u &= g \quad \text{on } \Gamma \times (0,T) \subset \mathbb{R}^3. \end{aligned}$$

Set $\Sigma := (0,L) \times (0,T) \subset \mathbb{R}^2$. For $\varphi, \psi \in C_0^\infty(\Sigma)$ it follows

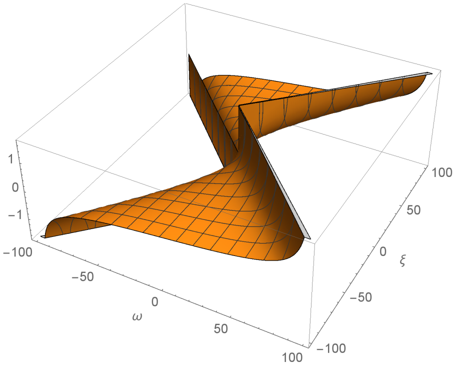
$$\begin{aligned} a_E(\varphi,\psi) =& \langle \partial_t (\mathcal{V}\varphi), \psi \rangle_\Sigma \\ =& \frac{1}{2} \int\limits_{\xi \in \mathbb{R}} \int\limits_{|\omega|>|\xi|} \frac{|\omega|}{\sqrt{\omega^2 - \xi^2}} \tilde{\varphi}(\xi,\omega) \overline{\tilde{\psi}(\xi,\omega)} \mathrm{d}\omega \mathrm{d}\xi \\ &+ \frac{\mathrm{i}}{2} \int\limits_{\xi \in \mathbb{R}} \int\limits_{|\omega|<|\xi|} \frac{\omega}{\sqrt{\xi^2 - \omega^2}} \tilde{\varphi}(\xi,\omega) \overline{\tilde{\psi}(\xi,\omega)} \mathrm{d}\omega \mathrm{d}\xi. \end{aligned}$$

Fourier-Fourier Symbol of $\partial_t \mathcal{V}$

Real part



Imaginary part



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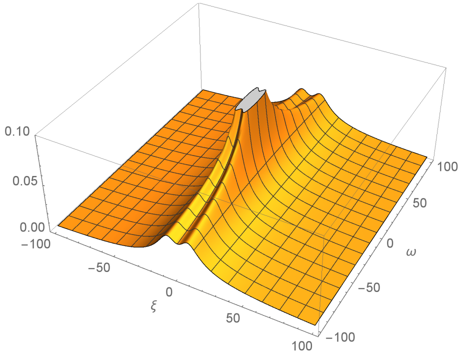
Fourier-Fourier Symbol of $\mathcal{V}$ for the heat equation

Real part



Fourier Representation

- It holds for all $\varphi, \psi \in L^2(0,T;H^{1/4}(\Gamma))$

$$|a_E(\varphi,\psi)| \leq C \cdot (1+T) \|\varphi\|_{L^2(0,T;H^{1/4}(\Gamma))} \|\psi\|_{L^2(0,T;H^{1/4}(\Gamma))}.$$

- For any $s \in \mathbb{R}$ there exists a sequence $(\varphi_n)_n$ of non-vanishing functions in $C_0^\infty(\Sigma)$ such that

$$\lim_{n\to\infty} \frac{a_E(\varphi_n,\varphi_n)}{\|\varphi_n\|^2_{H^s(\mathbb{R}^2)}} = 0.$$

Wave Equation

Consider for a bounded domain $\Omega \subset \mathbb{R}^d$

$$\begin{aligned} \partial_{tt} u - \Delta_x u &= f && \text{in } \Omega \times (0,T) =: Q, \\ u(\,\cdot\,, 0) &= 0 && \text{in } \Omega, \\ \partial_t u(\,\cdot\,, 0) &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \times [0,T] =: \Sigma. \end{aligned}$$

Integration by parts yields

$$\begin{aligned} \int_0^T \int_\Omega \partial_{tt} u \cdot v \mathrm{d}x \mathrm{d}t = &- \int_0^T \int_\Omega \partial_t u \cdot \partial_t v \mathrm{d}x \mathrm{d}t \\ &+ \int_\Omega \partial_t u(x,T) \underbrace{v(x,T)}_{\stackrel{!}{=}0} \mathrm{d}x - \int_\Omega \underbrace{\partial_t u(x,0)}_{=0} v(x,0) \mathrm{d}x. \end{aligned}$$

Wave Equation in $H^1(Q)$

Set

$$H^{1,1}_{0;0,}(Q) := L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;L^2(\Omega)) \subset H^1(Q),$$
$$H^{1,1}_{0;,0}(Q) := L^2(0,T;H^1_0(\Omega)) \cap H^1_{,0}(0,T;L^2(\Omega)) \subset H^1(Q).$$

Theorem (Ladyzhenskaya 1973, Teorema 3.2; OS, Z. 2018)

For $f \in L^2(Q)$ exists a unique solution $u \in H^{1,1}_{0;0,}(Q)$ of the variational formulation

Find $u \in H^{1,1}_{0;0,}(Q)$ such that

$$-\langle \partial_t u, \partial_t v \rangle_Q + \langle \nabla_x u, \nabla_x v \rangle_Q = \langle f, v \rangle_Q \quad \forall v \in H^{1,1}_{0;,0}(Q).$$

and it holds

$$|u|_{H^1(Q)} \leq \frac{1}{\sqrt{2}} T \|f\|_{L^2(Q)}.$$

Wave Equation in $H^1(Q)$

Eigenfunctions ϕ_i : $\quad -\Delta_x \phi = \mu\phi \;$ in Ω, $\quad \phi = 0$ on $\partial\Omega$

Ansatz: $u(x,t) = \sum_{i=1}^{\infty} U_i(t)\phi_i(x)$

Testfunction: $v(x,t) = V(t)\phi_j(x)$ for $V \in H^1_{,0}(0,T)$

Find $u \in H^{1,1}_{0;0,}(Q)$ such that

$$-\langle \partial_t u, \partial_t v\rangle_Q + \langle \nabla_x u, \nabla_x v\rangle_Q = \langle f, v\rangle_Q \quad \forall v \in H^{1,1}_{0;,0}(Q).$$

Find $U_j \in H^1_{0,}(0,T)$ such that

$$-\big\langle \partial_t U_j, \partial_t V\big\rangle_{(0,T)} + \mu_j \big\langle U_j, V\big\rangle_{(0,T)} = \big\langle f, V\phi_j\big\rangle_Q \quad \forall V \in H^1_{,0}(0,T).$$

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ODE corresponding to the Wave Equation

Consider for $\mu > 0$

$$\partial_{tt} u(t) + \mu u(t) = f(t) \quad \text{for } t \in (0,T), \quad u(0) = \partial_t u(0) = 0.$$

Variational formulation:
Find $u \in H^1_{0,}(0,T)$ such that

$$-\langle \partial_t u, \partial_t v \rangle_{(0,T)} + \mu \langle u, v \rangle_{(0,T)} = \langle f, v \rangle_{(0,T)} \quad \forall v \in H^1_{,0}(0,T),$$

where $f \in [H^1_{,0}(0,T)]'$ is given.

Theorem (OS, Z. 2018)

There exists a unique solution $u \in H^1_{0,}(0,T)$ satisfying

$$|u|_{H^1(0,T)} \leq \frac{2 + \sqrt{\mu} T}{2} \, \|f\|_{[H^1_{,0}(0,T)]'}.$$

Galerkin-Petrov Discretisation

Find $u_h \in S_h^1(0,T) \cap H^1_{0,}(0,T)$ such that

$$a(u_h, v_h) := -\langle \partial_t u_h, \partial_t v_h \rangle_{(0,T)} + \mu \langle u_h, v_h \rangle_{(0,T)} = \langle f, v_h \rangle_{(0,T)}$$

$\forall v_h \in S_h^1(0,T) \cap H^1_{,0}(0,T)$.

This method is „stable“ for $h < \sqrt{\frac{12}{\mu}}$.

Stabilisation [Zlotnik 1994]:

Find $u_h \in S_h^1(0,T) \cap H^1_{0,}(0,T)$ such that

$$a_h(u_h, v_h) := -\langle \partial_t u_h, \partial_t v_h \rangle_{(0,T)} + \mu \left\langle u_h, Q_h^0 v_h \right\rangle_{(0,T)} = \langle f, v_h \rangle_{(0,T)}$$

$\forall v_h \in S_h^1(0,T) \cap H^1_{,0}(0,T)$, where $Q_h^0 \colon L^2(0,T) \to S_h^0(0,T)$.

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Galerkin-Petrov Discretisation

Representation:

$$a(u_h, v_h) = \underbrace{-\langle \partial_t u_h, \partial_t v_h \rangle_{(0,T)} + \mu \langle u_h, Q_h^0 v_h \rangle_{(0,T)}}_{= a_h(u_h, v_h)} + \sum_{\ell=1}^{N} \frac{h_\ell^2 \mu}{12} \langle \partial_t u_h, \partial_t v_h \rangle_{\tau_\ell}$$

Lemma

For each $u_h \in S_h^1(0,T) \cap H_{0,}^1(0,T)$ holds the discrete inf-sup condition

$$c_{S,\mathrm{stab}}(T,\mu) |u_h|_{H^1(0,T)} \leq \sup_{0 \neq v_h \in S_h^1(0,T) \cap H_{,0}^1(0,T)} \frac{|a_h(u_h, v_h)|}{|v_h|_{H^1(0,T)}}$$

with the discrete inf-sup constant $c_{S,\mathrm{stab}}(T,\mu)$ for any mesh with maximal mesh size h.

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Galerkin-Petrov Discretisation

Find $u_h \in S_h^1(0,T) \cap H^1_{0,}(0,T)$ such that

$$a_h(u_h, v_h) := -\langle \partial_t u_h, \partial_t v_h \rangle_{(0,T)} + \mu \left\langle u_h, Q_h^0 v_h \right\rangle_{(0,T)} = \langle f, v_h \rangle_{(0,T)}$$

$\forall v_h \in S_h^1(0,T) \cap H^1_{,0}(0,T)$, where $Q_h^0 \colon\ L^2(0,T) \to S_h^0(0,T)$.

Theorem

Let the unique solution $u \in H^s(0,T)$ be with $s \in [1,2]$ and consider a mesh with maximal mesh size h. Then

$$|u - u_h|_{H^1(0,T)} \leq c(T,\mu,s)\, h^{s-1} \|u\|_{H^s(0,T)} + \mu\, h^2 (1 + \sqrt{2}\mu T^2) C\, |u|_{H^1(0,T)}$$

with a constant $c(T,\mu,s) > 0$ depending on T, μ, s and $C > 0$.

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Numerical Example without Stabilisation

Uniform decomposition of $(0, 10)$, $\mu = 1000$, $u(t) = \sin^2\left(\frac{5}{4}\pi t\right)$

N	h	$\|u - u_h\|_{L^2(0,10)}$	eoc	$\lvert u - u_h\rvert_{H^1(0,10)}$	eoc
4	2.5000000	7.0573e+01	0.00	9.8785e+01	0.00
8	1.2500000	1.6871e+03	-4.58	3.7166e+03	-5.23
16	0.6250000	9.1421e+07	-15.73	3.7247e+08	-16.61
32	0.3125000	2.3915e+15	-24.64	1.9496e+16	-25.64
64	0.1562500	1.6337e+22	-22.70	2.9536e+23	-23.85
128	0.0781250	3.1417e-02	78.78	1.7859e+00	77.13
256	0.0390625	9.2885e-03	1.76	8.2361e-01	1.12
512	0.0195312	2.4767e-03	1.91	3.9567e-01	1.06
1024	0.0097656	6.3105e-04	1.97	1.9532e-01	1.02
2048	0.0048828	1.5839e-04	1.99	9.7325e-02	1.00
4096	0.0024414	3.9633e-05	2.00	4.8620e-02	1.00
8192	0.0012207	9.9106e-06	2.00	2.4304e-02	1.00
16384	0.0006104	2.4778e-06	2.00	1.2152e-02	1.00
32768	0.0003052	6.1946e-07	2.00	6.0757e-03	1.00

Note that

$$h < \sqrt{\frac{12}{\mu}} \approx 0.1095.$$

Numerical Example with Stabilisation

Uniform decomposition of $(0,10)$, $\mu = 1000$, $u(t) = \sin^2\left(\frac{5}{4}\pi t\right)$

N	h	$\|u-u_h\|_{L^2(0,10)}$	eoc	$\|u-u_h\|_{H^1(0,10)}$	eoc
4	2.5000000	1.7722e+00	0.00	9.0867e+00	0.00
8	1.2500000	6.0704e+00	-1.78	2.0130e+01	-1.15
16	0.6250000	1.2687e+00	2.26	9.4204e+00	1.10
32	0.3125000	5.7861e+00	-2.19	6.0121e+01	-2.67
64	0.1562500	3.3966e-01	4.09	6.1941e+00	3.28
128	0.0781250	7.6647e-02	2.15	2.2955e+00	1.43
256	0.0390625	2.0315e-02	1.92	9.4091e-01	1.29
512	0.0195312	5.2649e-03	1.95	4.1539e-01	1.18
1024	0.0097656	1.3365e-03	1.98	1.9803e-01	1.07
2048	0.0048828	3.3682e-04	1.99	9.7671e-02	1.02
4096	0.0024414	8.4229e-05	2.00	4.8663e-02	1.01
8192	0.0012207	2.1057e-05	2.00	2.4310e-02	1.00
16384	0.0006104	5.2644e-06	2.00	1.2152e-02	1.00
32768	0.0003052	1.3161e-06	2.00	6.0758e-03	1.00

Note that

$$h < \sqrt{\frac{12}{\mu}} \approx 0.1095.$$

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Galerkin-Petrov Discretisation for the Wave Equation

Find $u_h \in Q_h^1(Q) \cap H^{1,1}_{0;0,}(Q)$ such that

$$-\langle \partial_t u_h, \partial_t v_h \rangle_Q + \sum_{m=1}^{d} \langle \partial_{x_m} u_h, \partial_{x_m} v_h \rangle_Q = \langle f, v_h \rangle_Q \quad \forall v_h \in Q_h^1(Q) \cap H^{1,1}_{0;,0}(Q).$$

This method is „stable“ for $h_t \leq \frac{1}{\sqrt{d}} h_x$.

Stabilisation [Zlotnik 1994]:
Find $u_h \in Q_h^1(Q) \cap H^{1,1}_{0;0,}(Q)$ such that

$$-\langle \partial_t u_h, \partial_t v_h \rangle_Q + \sum_{m=1}^{d} \langle \partial_{x_m} u_h, Q^0_{h_t} \partial_{x_m} v_h \rangle_Q = \langle f, v_h \rangle_Q \quad \forall v_h \in Q_h^1(Q) \cap H^{1,1}_{0;,0}(Q),$$

where $Q^0_{h_t} : \; L^2(0,T) \to S^0_{h_t}(0,T)$.

$L^2(Q)$ Stability and $L^2(Q)$ Error Estimate

Find $u_h \in Q_h^1(Q) \cap H^{1,1}_{0;0,}(Q)$ such that

$$-\langle \partial_t u_h, \partial_t v_h \rangle_Q + \sum_{m=1}^{d} \langle \partial_{x_m} u_h, Q^0_{h_t} \partial_{x_m} v_h \rangle_Q = \langle f, v_h \rangle_Q \quad \forall v_h \in Q_h^1(Q) \cap H^{1,1}_{0;,0}(Q).$$

Theorem

For $f \in L^2(Q)$ exists a unique solution $u_h \in Q_h^1(Q) \cap H^{1,1}_{0;0,}(Q)$ satisfying the stability estimate

$$\|u_h\|_{L^2(Q)} \leq 2T \|f\|_{[H^1_{,0}(0,T;L^2(\Omega))]'}$$

and the L^2 error estimate

$$\begin{aligned} \|u - u_h\|_{L^2(Q)} \leq\; & C\, h_x^2 \left(\|u\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t u\|_{L^2(0,T;H^2(\Omega))} \right) \\ & + C\, h_x\, h_t \|\partial_t \nabla_x u\|_{L^2(Q)} \\ & + C\, h_t^2 \left(\|\partial_{tt} u\|_{L^2(Q)} + \|\partial_{tt} \Delta_x u\|_{L^2(Q)} + \|\partial_t \Delta_x u\|_{L^2(Q)} \right). \end{aligned}$$

Numerical Example for $Q := (0,1) \times (0,10)$

$u(x,t) = \sin(\pi x)\sin^2\left(\frac{5}{4}\pi t\right)$:

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\|u-u_h\|_{L^2(Q)}$	eoc	$\|u-u_h\|_{H^1(Q)}$	eoc
30	0.37500	0.06250	3.75000	0.62500	3.579e+00	0.00	1.289e+01	0.00
132	0.18750	0.03125	1.87500	0.31250	1.975e+00	0.86	9.849e+00	0.39
552	0.09375	0.01562	0.93750	0.15625	9.213e-01	1.10	6.534e+00	0.59
2256	0.04688	0.00781	0.46875	0.07812	6.829e-01	0.43	5.210e+00	0.33
9120	0.02344	0.00391	0.23438	0.03906	2.466e-01	1.47	2.848e+00	0.87
36672	0.01172	0.00195	0.11719	0.01953	7.029e-02	1.81	1.435e+00	0.99
147072	0.00586	0.00098	0.05859	0.00977	1.819e-02	1.95	7.159e-01	1.00
589056	0.00293	0.00049	0.02930	0.00488	4.588e-03	1.99	3.576e-01	1.00
2357760	0.00146	0.00024	0.01465	0.00244	1.149e-03	2.00	1.788e-01	1.00
9434112	0.00073	0.00012	0.00732	0.00122	2.875e-04	2.00	8.938e-02	1.00
37742592	0.00037	0.00006	0.00366	0.00061	7.189e-05	2.00	4.469e-02	1.00

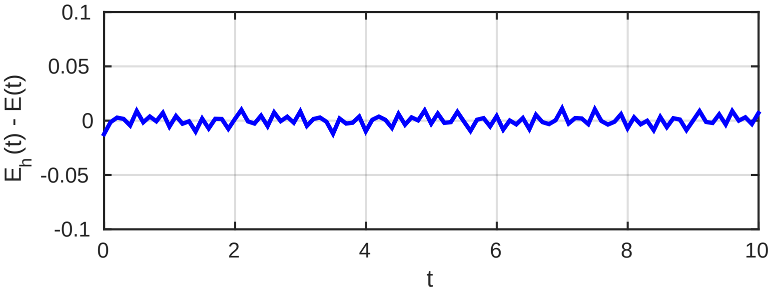
$u(x,t) = \sin(\pi x) t^2 (10-t)^{3/4}$:

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\|u-u_h\|_{L^2(Q)}$	eoc	$\|u-u_h\|_{H^1(Q)}$	eoc
30	0.37500	0.06250	3.75000	0.62500	7.836e+01	0.00	3.173e+02	0.00
132	0.18750	0.03125	1.87500	0.31250	2.166e+01	1.86	1.191e+02	1.41
552	0.09375	0.01562	0.93750	0.15625	5.487e+00	1.98	5.225e+01	1.19
2256	0.04688	0.00781	0.46875	0.07812	1.777e+00	1.63	2.696e+01	0.95
9120	0.02344	0.00391	0.23438	0.03906	6.476e-01	1.46	1.593e+01	0.76
36672	0.01172	0.00195	0.11719	0.01953	3.001e-01	1.11	1.076e+01	0.57
147072	0.00586	0.00098	0.05859	0.00977	1.393e-01	1.11	8.077e+00	0.41
589056	0.00293	0.00049	0.02930	0.00488	6.156e-02	1.18	6.452e+00	0.32
2357760	0.00146	0.00024	0.01465	0.00244	2.650e-02	1.22	5.308e+00	0.28
9434112	0.00073	0.00012	0.00732	0.00122	1.126e-02	1.23	4.423e+00	0.26
37742592	0.00037	0.00006	0.00366	0.00061	4.758e-03	1.24	3.704e+00	0.26

Energy Conservation for $Q := (0,1) \times (0,10)$

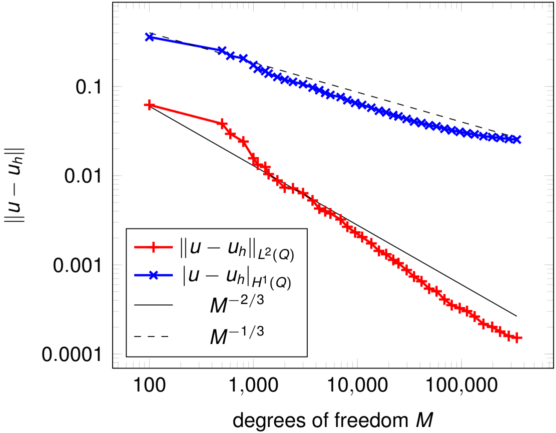
$u(x,t) = (\cos(\pi t) + \sin(\pi t)) \sin(\pi x)$:

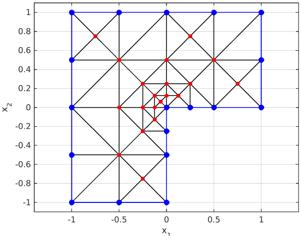
- $f = 0$
- $E(t) := \frac{1}{2}\|\partial_t u(\cdot,t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla_x u(\cdot,t)\|^2_{L^2(\Omega)} = \frac{\pi^2}{2}$ for $t \in [0,T]$

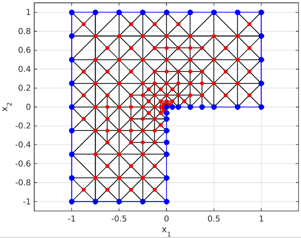


Numerical Example for $Q := \text{L-Shape} \times (0,1)$

$u(x_1, x_2, t) = \sin(\pi t) r^{2/3} \sin(2/3\omega)$, time mesh size $h_t = 0.01$







Summary

- Boundary integral equations
- Second-order ODE for $f \in [H^1_{,0}(0,T)]'$
  - „Stability“ for $h < \sqrt{\frac{12}{\mu}}$
  - No condition with stabilisation
  - Error estimates
- Wave equation for $f \in L^2(Q)$
  - Tensor product ansatz
  - CFL condition $h_t \leq \frac{1}{\sqrt{d}} h_x$
  - No CFL condition with stabilisation
  - $L^2(Q)$ stability
  - $L^2(Q)$ error estimates

Accepted for publication:

- A Stabilized Space–Time Finite Element Method for the Wave Equation

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